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ON THE REGULAR PRECESSION OF A BODY OF REVOLUTION ON A HORIZONTAL PLANE WITH FRICTION*

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Conditions of existence and stability of regular precession of a heavy body of revolution on a horizontal plane with friction are determined. Obtained results are compared with those of similar investigations of regular precession of bodies of revolution on smooth and rough surfaces.

1. Consider a dynamically symmetric heavy solid body bounded by a convex surface of revolution and supported by a horizontal plane. The latter exerts on the body at the contact point, besides the normal reaction, a force proportional to the contact point velocity in a direction opposite to the latter (the viscous friction force).

The position of the body is defined in a fixed coordinate system $Oxyz$ by the coordinates x and y of its center of mass (plane Oxy is the supporting plane and the Oz axis is directed vertically upward) and by Euler's angles θ, φ and ψ between the principal central axes $G\xi$, $G\eta$ and $G\zeta$ of inertia of the body and the axes of the fixed coordinate system. Assuming the body center of mass to be on its dynamic symmetry axis, which is also the axis of the body surface symmetry, and directing the $G\zeta$ axis along the latter, we obtain for the Lagrangian function and the Rayleigh dissipative function expressions of the form

$$L=\frac{1}{2}(A+m\varkappa^2)\theta^{\cdot 2}+\frac{1}{2}C\varphi^{\cdot 2}+\frac{1}{2}(A\sin^2\theta+C\cos^2\theta)\psi^{\cdot 2}+$$
$$C\varphi^{\cdot}\psi^{\cdot}\cos\theta+\frac{1}{2}m(x^{\cdot 2}+y^{\cdot 2})-mgh$$
$$F=\frac{1}{2}mk[(x^{\cdot}-\alpha_1\theta^{\cdot}-\alpha_2\varphi^{\cdot}-\alpha_3\psi^{\cdot})^2+(y^{\cdot}-\beta_1\theta^{\cdot}-\beta_2\varphi^{\cdot}-\beta_3\psi^{\cdot})^2]$$
$$\alpha_1=h\sin\psi,\ \ \alpha_2=\chi\cos\psi,\ \ \alpha_3=\varkappa\cos\psi$$
$$\beta_i=-\partial\alpha_i/\partial\psi\ (i=1,2,3)$$
$$h=-(\chi\sin\theta+\zeta\cos\theta),\ \ \varkappa=\chi\cos\theta-\zeta\sin\theta,$$
$$\chi=\xi\sin\varphi+\eta\cos\varphi$$

where m is the mass of the body, A and C are, respectively, the central equatorial and the axial moments of inertia, $k>0$ is the friction coefficient, ξ, η, ζ are coordinates of the point of body contact with the plane in the coordinate system $G\xi\eta\zeta$, and h is the height of the body center of mass above the support plane. It can be shown that χ and ζ are functions of the single variable θ which are determined by the form of the equation that defines the surface bounding the body and satisfy the relation

$$\frac{d\chi}{d\theta}\sin\theta+\frac{d\zeta}{d\theta}\cos\theta=0 \tag{1.1}$$

Since function F expressed in terms of initial variables explicitly depends on ψ, we can substitute for the x and y coordinates the quasi-coordinates ρ and σ defined by formulas

$$\rho^{\cdot}=x^{\cdot}\sin\psi-y^{\cdot}\cos\psi,\quad \sigma^{\cdot}=x^{\cdot}\cos\psi+y^{\cdot}\sin\psi \tag{1.2}$$

where $\sigma^{\cdot}$ is the projection of velocity of the body center of mass on the line of nodes, and $\rho^{\cdot}$ is its projection on the orthogonal to the latter in the horizontal plane.

We denote functions L and F after the elimination in them of $x^{\cdot}$ and $y^{\cdot}$ using formulas (1.2), by L^* and F^* which depend only on θ, $\theta^{\cdot}$, $\varphi^{\cdot}$, $\psi^{\cdot}$, $\rho^{\cdot}$, $\sigma^{\cdot}$. In particular

$$F^*={}^1/_2mk[(\rho^{\cdot}-h\theta^{\cdot})^2+(\sigma^{\cdot}-\chi\varphi^{\cdot}-\varkappa\psi^{\cdot})^2]$$

The equations of motion in new variables

$$\frac{d}{dt}\frac{\partial L^*}{\partial\theta^{\cdot}}=\frac{\partial L^*}{\partial\theta}-\frac{\partial F^*}{\partial\theta^{\cdot}},\quad \frac{d}{dt}\frac{\partial L^*}{\partial\varphi^{\cdot}}=-\frac{\partial F^*}{\partial\varphi^{\cdot}},\quad \frac{d}{dt}\frac{\partial L^*}{\partial\psi^{\cdot}}=-\frac{\partial F^*}{\partial\psi^{\cdot}} \tag{1.3}$$

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$$\frac{d}{dt}\frac{\partial L^*}{\partial \rho^\cdot} = -\frac{\partial F^*}{\partial \rho^\cdot} + \frac{\partial L^*}{\partial \sigma^\cdot}\psi^\cdot, \quad \frac{d}{dt}\frac{\partial L^*}{\partial \sigma^\cdot} = -\frac{\partial F^*}{\partial \sigma^\cdot} - \frac{\partial L^*}{\partial \rho^\cdot}\psi^\cdot$$

do not explicitly depend on φ, ψ, ρ and σ, containing only their velocities and accelerations (such variables are called quasi-cyclic /1/ or pseudo-cyclic /2/, and sometimes simply cyclic /3/, (in the sense of being ignorable)). Hence it is reasonable to state the problem of steady motions of system (1.3) as

$$\theta = \theta_0, \quad \theta^\cdot = 0, \quad \varphi^\cdot = \varphi_0^\cdot, \quad \psi^\cdot = \psi_0^\cdot, \quad \rho^\cdot = \rho_0^\cdot, \quad \sigma^\cdot = \sigma_0^\cdot \tag{1.4}$$

The substitution of (1.4) into the equations of motion (1.3) taking into account (1.1), shows that solution (1.4) obtains when the constants $\theta_0, \varphi_0^\cdot, \psi_0^\cdot, \rho_0^\cdot, \sigma_0^\cdot$ satisfy the system of equations

$$\begin{aligned} &(A - C)\chi \sin\theta\cos\theta\,\psi^{\cdot 2} - C\sin\theta\varphi^\cdot\psi^\cdot + mg\varkappa + mh\sigma^\cdot\psi^\cdot = 0 \\ &\chi\rho^\cdot\psi^\cdot = 0, \quad \varkappa\rho^\cdot\psi^\cdot = 0, \quad -k\rho^\cdot + \sigma^\cdot\psi^\cdot = 0 \\ &k(\sigma^\cdot - \chi\varphi^\cdot - \varkappa\psi^\cdot) + \rho^\cdot\psi^\cdot = 0 \end{aligned} \tag{1.5}$$

2. Let us assume that the quantity θ_0 is such that

$$\chi(\theta_0) \neq 0, \ \varkappa(\theta_0) \neq 0 \tag{2.1}$$

since otherwise, as can be readily shown (see also formulas (2.4) and (2.5) below) that a regular precession of the body is impossible. Then from system (1.5) follows that $\rho_0^\cdot = \sigma_0^\cdot = 0$ and that the three constants $\theta_0, \varphi_0^\cdot = \Omega, \psi_0^\cdot = \omega$ satisfy the system of two equations

$$\begin{aligned} &(A - C)\sin\theta_0\cos\theta_0\omega^2 - C\sin\theta_0\Omega\omega + mg\varkappa_0 = 0 \\ &\chi_0\Omega + \varkappa_0\omega = 0 \end{aligned} \tag{2.2}$$

here and everywhere below, the zero subscript indicates that the respective function of variable θ is calculated for $\theta = \theta_0$). The respective one-parameter set of solutions of the form

$$\theta = \theta_0, \ \theta^\cdot = 0, \quad \varphi^\cdot = \Omega, \ \psi^\cdot = \omega, \ \rho^\cdot = \sigma^\cdot = 0 \tag{2.3}$$

then defines a regular precession of the body.

Indeed, under conditions (2.3) the body center of mass is stationary, the body rotates at constant angular velocity ω about the vertical passing through its center of mass, and at constant angular velocity Ω about its own axis of symmetry which is at angle θ_0 to the vertical. The point of contact of the body with the plane describes two circles, viz. one of radius $|\varkappa_0|$ and center at the projection of the body center of mass on the supporting plane, at angular velocity ω, and the second at angular velocity Ω on the body surface in a plane orthogonal to its axis of symmetry a circle of radius $|\chi_0|$ and center at the coordinate ζ_0 on that plane. By virtue of the second equation of system (2.2) the body regular precession on a plane with friction occurs without slip.

Expressing Ω in terms of ω defined in the second equation of system (2.2) and substituting the obtained value of Ω into the first equation, we obtain

$$\Omega = -\left(\frac{\varkappa}{\chi}\right)_0\omega, \quad \omega^2 = -\left[\frac{mg\chi\varkappa}{(A\chi\cos\theta - C\zeta\sin\theta)\sin\theta}\right]_0 \tag{2.4}$$

For a given inclination of the body axis of symmetry from the vertical formulas (2.4) uniquely define the absolute values of angular velocities of precession and of the proper rotation of the body. Moreover, the second of these provides the condition of existence of regular precession of a body on a plane with friction (since $\omega^2 > 0$), whose axis of symmetry is at angle θ_0 to the vertical

$$(Al_0\cos\theta_0 + C\zeta_0)(l_0\cos\theta_0 + \zeta_0) > 0; \quad l = -\chi/\sin\theta \tag{2.5}$$

Note that when this condition is satisfied, formulas (2.1) are automatically satisfied. Function $\chi(\theta)$ has been replaced here by function $l(\theta)$ representing the distance from the contact point of the body and plane to the point of intersection of the body axis of symmetry with the vertical passing through the first point.

In the cases of absolutely smooth and absolutely rough horizontal planes regular precessions of the body form two-parameter sets /4,5/. In the first of these cases the three constants θ_0, Ω and ω satisfy a single equation that coincides with the first equation of system (2.2), and in regular precession, as in this case, the center of mass is stationary, but the body can slip on the supporting plane /4/; in the second case the constants θ_0, Ω and ω also satisfy a single equation which is obtained from the first equation of system (2.2), by substituting

$$-mh_0(\varkappa_0\omega+\chi_0\Omega)\omega \tag{2.6}$$

for the zero in the right-hand side of the latter.

Slipping of the body is not possible, and in regular precession the stationary point of the body is on its axis of symmetry at the distance

$$|(\varkappa_0\omega+\chi_0\Omega)/(\omega\sin\theta_0)| \tag{2.7}$$

from its center of mass /5/.

Comparison of the second equation of system (2.2) with formulas (2.6) and (2.7) shows that the one-dimensional manifold of regular precessions of a heavy body of revolution on a plane with friction lies at the intersection (nonempty under condition (2.5)) of respective two-dimensional manifolds in the cases of smooth and rough planes.

Remarks. 1°. If θ_0 in (1.4) is such that $\chi_0\neq 0$, $\varkappa_0=0$, then in the case of $\theta=\theta_0$ the body center of mass lies on the vertical passing through the point of its contact with the supporting plane (as in the position of equilibrium). Then if follows from system (2.2) that $\psi_0^{\cdot}=\rho_0^{\cdot}=0$, $\sigma_0^{\cdot}=\chi_0\varphi_0^{\cdot}$, and the respective one-parameter set of solutions of the form

$$\theta=\theta_0,\quad \theta^{\cdot}=0,\quad \varphi^{\cdot}=\Omega,\quad \psi^{\cdot}=\rho^{\cdot}=0,\quad \sigma^{\cdot}=\chi_0\Omega$$

where $(\varphi_0^{\cdot}\equiv\Omega$ is arbitrary, defines the rolling of the body at constant velocity along a fixed straight line (for $\Omega=0$ we have equilibrium). Similar conditions define the rolling of a body on smooth and rougth surfaces /4,5/.

2°. If θ_0 in (1.4) is such that $\chi_0=0$, then for $\psi_0^{\cdot}\neq 0$ we have permanent rotation of the body about its vertically directed axis of symmetry (for $\psi_0^{\cdot}=0$ we have equilibrium), and the respective results follows as a particular case from /6/, where the conditions of existence and stability of permanent rotations of an arbitrary body on a plane with friction were derived.

3. Let us investigate the stability of regular precession of a body (of solution (2.3)) with respect to perturbations of variables θ, $\theta^{\cdot}$, $\varphi^{\cdot}$, $\psi^{\cdot}$, $\rho^{\cdot}$ and $\sigma^{\cdot}$. Setting

$$\theta-\theta_0=u,\ \varphi^{\cdot}+\psi^{\cdot}\cos\theta_0-(\Omega+\omega\cos\theta_0)=v$$
$$\psi^{\cdot}-\omega=w,\quad \rho^{\cdot}=\rho',\ \sigma^{\cdot}=\sigma'$$

we reduce the equations of perturbed motion to the form

$$\begin{aligned}
&(A+m\mu_0^2\sin^2\theta_0)u^{\cdot\cdot}=[-(A-C)\omega^2\sin^2\theta_0+mg(l_0-\\
&\quad r_0)]u-C\omega\sin\theta_0 v+[(2A-C)\omega\cos\theta_0-C\Omega]\sin\theta_0 w-\\
&\quad m\lambda_0(\rho'^{\cdot}-\omega\sigma')+U\\
&Cv^{\cdot}=C\omega\sin\theta_0 u^{\cdot}+ml_0\sin\theta_0(\omega\rho'+\sigma'^{\cdot})+V\\
&A\sin^2\theta_0 w^{\cdot}=-[(2A-C)\omega\cos\theta_0-C\Omega]\sin\theta_0 u^{\cdot}+\\
&\quad m\zeta_0\sin\theta_0(\omega\rho'+\sigma'^{\cdot})+W\\
&\rho'^{\cdot}=k\lambda_0 u^{\cdot}-k\rho'+\omega\sigma'+R\\
&\sigma'^{\cdot}=k[(\lambda_0-r_0)\omega-r_0\Omega\cos\theta_0]u-kl_0\sin\theta_0 v-k\zeta_0\times\\
&\quad\sin\theta_0 w-\omega\rho'-k\sigma'+S\\
&(\lambda_0=l_0\sin^2\theta_0-\zeta_0\cos\theta_0,\ \mu_0=l_0\cos\theta_0-\zeta_0)
\end{aligned}\tag{3.1}$$

where $r_0=r(\theta_0)$ in which $r(\theta)$ is the radius of curvature of the body surface meridian cross section at the contact point of the body and support plane, and U, V, W, R and S are functions of variables u, $u^{\cdot}$, v, w, ρ' and σ' whose expansions in powers of the indicated variables begin with terms of order of smallness not lower than the second. Setting these functions equal zero, we obtain a linearized system of equations of the perturbed motion, which admits the linear integral

$$2(Al_0\cos\theta_0+C\zeta_0)\omega\sin\theta_0 u-C\zeta_0 v+Al_0\sin^2\theta_0 w=\pi=\text{const} \tag{3.2}$$

and whose characteristic equation has always one zero root.

If for the variable w using formula (3.2) we substitute the variable π which defines the deviation of perturbed motion from solution (2.3) along the manifold (2.2), and for variables u and v substitute in the manner shown in /7/ the variables u' and v' that define that deviation from manifold (2.2), and rewrite system (3.1) in the new variables u', $u^{\cdot}$, v', π, ρ' and σ', then all nonlinearities of derived equations identically vanish when $u'=u^{\cdot}=v'=\rho'=\sigma'=0$.

Thus, if all roots of the characteristic equation, except the zero one, of the linearized system of equations of perturbed motion lie in the left-hand half-plane, we have the particular case of the single zero root, and the Liapunov theorem /7/ applies. The regular precession of a body of revolution on a plane with friction is then (unlike in the case of smooth and rough planes) asymptotically stable with respect to a part of variables that define the deviation of perturbed motion from the regular precessions manifold.

Omitting owing to unwieldiness the presentation in explicit form of coefficients of the indicated characteristic equation

$$\lambda\,(a_0\lambda^5 + a_1\lambda^4 + a_2\lambda^3 + a_3\lambda^2 + a_4\lambda + a_5) = 0 \quad (a_0 > 0,\ a_1 > 0)$$

and of conditions /8/

$$a_3 > 0, \quad a_5 > 0, \quad a_1a_2 - a_0a_3 > 0$$
$$(a_1a_2 - a_0a_3)(a_3a_4 - a_2a_5) - (a_0a_5 - a_1a_4)^2 > 0$$

under which five of its zero roots lie in the left-hand half-plane. Note that when $k = 0$ that equation assumes the form

$$\lambda^2\,(\lambda^2 + \omega^2)(a_*\lambda^2 + b_*) \quad (a_* > 0)$$

and when $k = \infty$

$$\lambda^2\,(a_{**}\lambda^2 + b_{**}) \quad (a_{**} > 0)$$

Conditions $b_* > 0$ and $b_{**} > 0$ match the conditions of stability of regular precession of a body of revolution on smooth or rough surfaces, respectively.

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